~~Current~~

Breaking E_8 is subtle —

~ −9, −10, −11 curve in 6D, E_8 w/ codim 2 (4,6) sings
→ strongly coupled / SCFT matter.

some progress: Y Wang + Tian.

Seems $E_8 \to E_7 \to$ GSM may work & be most "natural" SM realization.

Big issue: <u>What is F-theory?</u>

String theory: has world-sheet quantization.
in principle, SFT ⇒ complete bg-independent formulation,
can compute low-E EFT, corrections in pert expansion

M-theory: M(atrix) model (BFSS, dHN) ~ quantum thy in light cone

But F-theory has no real definition as complete theory.

Some understanding from

- Holomorphy / alg geometry ⇒ strong global picture
- Limit of M-theory gives some info — but singular ECY
  don't have formulation of theory on singular space
- IIB sugra, Sen (~ orbifold) limit
- Heterotic duality when B is $\mathbb{P}^1$ fibered
- String junctions
- Special cases: const τ

But we need more: IDEAS?

Approaches to realizing the Standard Model

	GUT	$SU(3)\times SU(2)\times U(1)\,(/\mathbb{Z}_6)$
Tuned G	Tuned GUT, eg. SU(5)	Tuned G_{SM}
Rigid G	rigid GUT	rigid G_{SM}

- Tuned SU(5): much work [BHV, DW]
- Tuned G_{SM}: can tune directly - "F_{11}" fiber [Klevers et al], direct constructn [Raghu/WT/Turner]
  "quadrillion SM" [Cvetic et al]

But: tuned models are rare: involve fine tuning,
many bases will not support

- SU(3) × SU(2) can be rigid / geometrically non-Higgsable in 4D
  U(1) factors difficult to integrate [Y Wang paper w/ Morrison]

Most natural approach: rigid GUT!

New ingredient in 4D: fluxes

- from G_{IJKL} in M-theory $G = dC$
- G ⇒ can break gauge gp.
- G ⇒ can induce chiral matter
- G ⇒ superpotential $W = \int \Omega_4 \wedge G_4$ [GVW]

Recent work w/ SY(Kobe) Li:
Break $E_7, E_6 \to G_{SM}$ w/ fluxes...

- some nice phenomenological properties (dim 4,5 proton decay suppressed by residual E_7 symmetries)
- Requires non-toric base.

5. 4D $\mathcal{N} = 1$ SUGRA

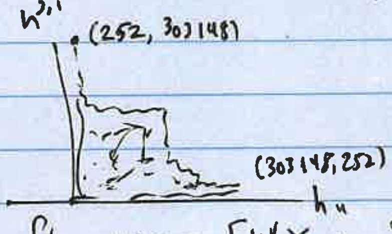
Many string constructions – huge literature!
Biggest issue: Moduli space lifted by potential/superpotential

$\mathrm{IIA}, \mathrm{IIB}/_{CY3 + \text{fluxes}}$ $\quad$ $\text{het}/_{CY3}$ $\quad$ $M/_{G2}$ $\quad$ $\text{F-theory}/_{B_3}$ base for ECY4.

Focus on F-theory. [Timo Weigand l. notes for more details]

$\{B_3, \text{ECY4's}\}$ vast, much less understood[B]

- Minimal model (Mori theory) program much more complicated, no simple analogue of just $\mathbb{P}^2$, $\mathbb{F}_m$
- Toric bases: $\sim 10^{750}$ explicit [Halv/Long/Sun]
  $\sim 10^{3000}$ by monte Carlo [w/ Y Wang]
- Similar big picture(?)



- $B_{\max h_{11}} \to \sim 10^{273,000}$ flux vacua [w/ Y Wang]
- $B_{\max h_{11}} \to \sim 10^{46,000}$ flop phases.
- Ignoring flop phases, $\sim 10^{90}$ diff polytopes [w/ Y Wang + Y. Yu]? (preliminary, but includes some codim 2 (4,6) loci

? What is the right ensemble?
? What is the measure?

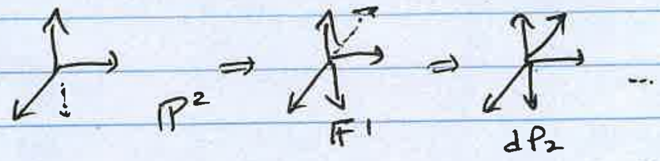
Unclear, but for any of these,
- many B_3, X_4
- almost all have $E_8, F_4, G_2 \times SU(2)$ + some other rigid gps.

Thm (Grassi / Gross)

All B_2 supporting ECY3's are blowups of $\mathbb{P}^2$, $\mathbb{F}_m$.

blow up torically: add ray $r' = r_i + r_{i+1}$

e.g.



$\Rightarrow$ all toric B_2's (~ 65k [MT]) max $h_{11}(B) = 193$

Similarly, can get non-toric (Wang w/ T. down to $h^{21} =$)

$\Rightarrow$ Gives many 6D F-theory models (+ tuning)

ECY3's.

KS database: Start w/ toric 3D reflexive polytope (convex hull of rays ignore Δ (no int pts, polar dual lattice polytope w/o int pts))

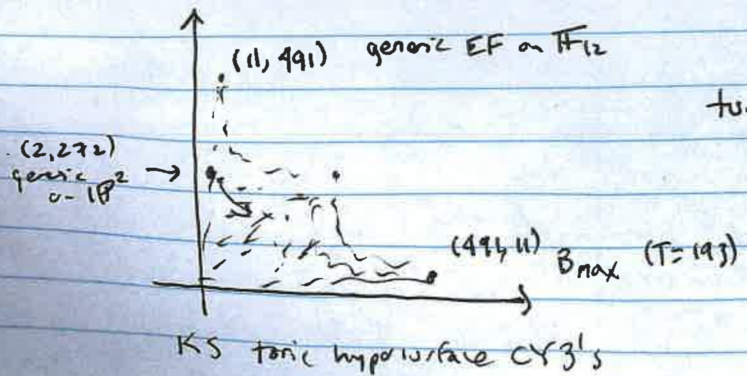
construct $-K$ hypersurface ($-K = \Sigma$ rays again)

$\rightarrow$ 4 M construction.

All but ~2,000 have "obvious" elliptic fiber (2D subpoly)

$\Rightarrow$ vast range of constructions.

Big picture of 6D SUGRA landscape



all connected

tunings:

- Tensor (blowup) transitions Δ
- Higgsing transitions ΔG
- "Matter transitions" ΔM

Discrete G, e.g $\mathbb{Z}_2$, from $U(1)$ w $q=1,2$
Higgs on $q=2$.

Captured in Tate-Shafarevich / Weil-Chatelet gp [Bouq/Mor ...]
— related to $g=1$ fibers (no sectn)

4.5 Classificat of 6D SUGRA + ECY3's.

Can we classify ("make a list of") 6D SUGRA?

A) F-theory: i) choose B_2
ii) look @ tuning G_{NA}, G_A, G_{dr}...

B) Look @ CY3's.
- CICY's
- Kreuzer - Skarke database (~400 M constructns)

Useful tool: toric geometry.
Combinatorial alg. geometry.
Rays in $\mathbb{Z}^n \longleftrightarrow$ divisors in X_n (not CY)

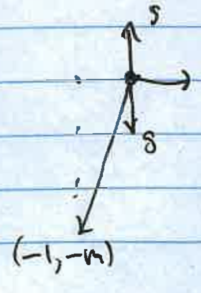
see eg. [Fulton]

Simple case: smooth 2D cpt toric vars.
rays r_i, r_i, r_{i+1} span unit cell
$(\det \begin{matrix} r_i \\ r_{i+1} \end{matrix} = \pm 1)$

ex.


$D_i \cdot D_{i+1} = 1$
$D_i \cdot D_j = 0$, non-adjacent.
$\sum r_i^x D_i = \sum r_i^y D_i = 0$ (SR ideal)
~~$\sum$ ... $D_i \cdot D_{i+1} + D_i \cdot D_{i-1} = 0$, $D_i \cdot D_i = m$~~
$-mD_i + D_{i+1} + D_{i-1} = 0$, $D_i \cdot D_i = -m$

$-K = \sum D_i$



$\mathbb{F}_m$

$-K = S + \tilde{S} + F + F$
$\tilde{S} = S + mF \Rightarrow -K = 2S + (2+m)F$

Duality w/ heterotic $\leftrightarrow$ $\mathbb{F}_m$ F-theory

het / K3, K3 elliptic $\bigcirc \leftrightarrow \bigcirc$

fibering 8D duality.

het bundle: need instantons $\eta_+ + \eta_- = 24$

Duality: $\eta_\pm = 12 \pm m$

e.g. $m = -12$, $\eta_+ = 0 \Rightarrow E_8$ unbroken, matches n.h. E_8.

[more: MV, FMW etc]

4.4 general structure: map 6D $\longleftrightarrow$ geometry

$T = h_{11}(B) - 1$

$H_{unch} = h_{2,1}(X) + 1$

$a = K$

$b_i = D_i$ divisor supporting G_i

6D SUGRA $\longleftrightarrow$ Elliptic CY3's.

Close match but still some "swampland" disagreement.

More subtle features:

U(1) factors: (Mordell-Weil gp) rk $\longrightarrow$ rational sections.

nonlocal, difficult to extract.

Shioda-Tate-Wazir: $h_{11}(X) = \underbrace{h_{11}(B)}_{\text{base}} + \underbrace{1}_{\text{fiber}} + \underbrace{rk(G)}_{\text{incl } G_A}$

U(1) matter subtle:

$\exists$ ∞ families w/ anom cancellation [TT]

problematic [RTT, CT]

general U(1) tuning Mor.-Park

$U(1)^2$ CKT

some $U(1)^3$

beyond harder

Ex. B = Hirzebruch $\mathbb{F}_m$

$\mathbb{F}_m$: $\mathbb{P}^1$ bundle over $\mathbb{P}^1$

Cpt of $\mathcal{L}$ with $[(f)] = \pm m$

$h_{11}(\mathbb{F}_m) = 2$. basis S (section), F = fiber
$\Rightarrow \boxed{T = 1}$

Intersection form: $S \cdot S = -m$, $F \cdot F = 0$, $S \cdot F = 1$
(other section: $\tilde{S} = S + mF$: $\tilde{S} \cdot \tilde{S} = m$)

$-K_{\mathbb{F}_m} = 2S + (2+m)F$ (compute later)

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$: $f_{8,8}$, $g_{12,12}$ in u, v

$a = K$ again, $a^2 = K^2 = (2S + 2F)^2 = 8 = 9 - T$

Interesting cases: $m \geq 3$

$\mathbb{F}_3$: $-K \cdot S = (2S + 5F) \cdot S = -1$! $\Rightarrow$ section of $-nK$ vanishes on S.

$-4K \cdot S = -4 \Rightarrow -4K = 2S + X$
$-6K \cdot S = -6 \Rightarrow -6K = 2S + Y$ effective

$\Rightarrow$ f, g vanish to ord $2, 2$ on S. $\Rightarrow$ $SU(3)$ <u>forced</u>
rigid / non-Higgsable gauge group.

No matter! $\tilde{g} \in -6K - 2S = 12S + 30F - 2S = 10(S + 3F) = 10\tilde{S}$,
$\tilde{g} \cdot S = 0$

generally
-3 curve	$\rightarrow$	NH $SU(3)$
-4	$\rightarrow$	$SO(8)$
-5	$\rightarrow$	F_4 (monodromy)
-6	$\rightarrow$	E_6
-7	$\rightarrow$	$E_7 + \frac{1}{2}$ 56
-8	$\rightarrow$	E_7
-9 – -12	$\rightarrow$	E_8
-13	$\rightarrow$	bad codim 1 sing.

4.2 Realizations through string theory

$$\begin{array}{ccccc} \text{het}/I & & \text{F-theory}/\text{ECY3} & & \text{IIB}/K3+D7 \ (\text{IBM}) \\ \downarrow K3 & & \downarrow & & \downarrow \\ 6D & & 6D & & 6D \\ (T=1) & \xleftrightarrow[\text{dual}]{} & (\text{any } T) & \xleftrightarrow[\text{dual}]{} & (T=1) \end{array}$$

6D F-theory $\quad \mathbb{E} \longrightarrow X = CY3$
$$\downarrow$$
B_2 cpx compact base surface

4.3 Examples of 6D F-theory

• $B = \mathbb{P}^2$. $\quad$ similar to $\mathbb{P}^1$.

$\mathbb{P}^2 \qquad -K(\mathbb{P}^2) = 3H$

Generic model: $f_{12} \in \mathcal{O}(-4K)$, $\quad g_{18} \in \mathcal{O}(-6K)$

generically, no forced G.

$T = 0 \qquad$ (IIB: D_4^+ wrapped on $H_{1,1}$ cycle $\to B^{\pm}$
$\Rightarrow T = h_{11}(B) - 1$)

$V = 0$

$\Rightarrow \quad H - V = 273 - 29T \Rightarrow H = 273$ (272 in parameters of f, g $\to$ cs moduli, 1 universal ~ axiodil)

Can tune G_{NA}, $\Rightarrow$ matter

e.g. E_6 on H: $\quad f = u^3 \tilde{f}_9, \quad g = u^4 \tilde{g}_{14}, \quad \Delta = u^8(\tilde{g}_{14}^2 + 4u$

adds node

Kac-Vafa
• xforms under $\underline{27}$

Note: $\tilde{g}_{14}(u,v)$ vanishes @ 14 pts on $u=0$.

$\Rightarrow 14 \times (\frac{1}{2}\underline{27})$ $\quad$ check eg. $\quad a \cdot b = \frac{1}{6}\lambda\,(A_{adj} - x_R^i A_R)$
(fund) rep $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (6)$

$a = K = -3, \ b = 1 \Rightarrow -3 = (4 - 7 \cdot 1)$ ✓

4. 6D $\mathcal{N}=(1,0)$ SUGRA

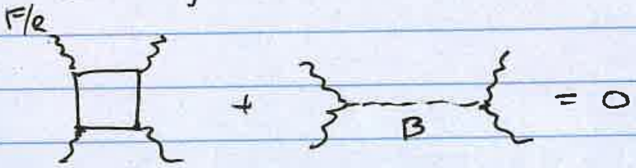
4.1 SUGRA multiplets

SUGRA	$(g,\ B^+_{\mu\nu},\ \psi^-_\mu)$
tensor (T)	$(B^-_{\mu\nu},\ \phi,\ \chi^+)$
vector (V)	$(A_\mu,\ \lambda^-)$
hypes (H)	$(4\varphi,\ \psi^+)$

(gauge gp $G = G^{NA} \times G^A / \Gamma$)
(matter in rep R of G)

like 10D, ($6 = 2R + 4k$)

Strong anomaly constraints



only NA, ($\sim b^{ij}$ for abelian)

$$\mathcal{L} \supset a_\alpha B^\alpha \wedge R \wedge R + b^i_\alpha B^\alpha \wedge F_i \wedge F_i$$

R^4: $\quad H - V = 273 - 29T$

$(R^2)^2$: $\quad a \cdot a = 9 - T$

[Sagnotti; KMT]

+ constraints on matter reps via gp thy.

F^4: $\quad 0 = B^i_{adj} - \sum_R \underset{\uparrow}{X^i_R} B^i_R$

\# reps in rep R of G_i

F^2R^2: $\quad a \cdot b_i = \frac{1}{6}\lambda^i \left(A^i_{adj} - \sum x^i_R A^i_R\right)$

$(F^2)^2$: $\quad b_i \cdot b_i = \frac{1}{3}\lambda^i \left(\sum x^i_R C_R - C^i_{adj}\right)$

$F_i^2 F_j^2$: $\quad b_i \cdot b_j = 2\sum x^{ij}_{R,S} A^i_R A^j_S \qquad i \neq j$

$$\mathrm{Tr}_R F^4 = B_R \,\mathrm{tr}\, F^4 + C_R (\mathrm{tr}\, F^2)^2$$
$$\mathrm{Tr}_R F^2 = A_R \underset{\uparrow \square}{\mathrm{tr}} F^2$$

$\lambda_{SU(N)} = 1, \quad \lambda_{E_8} = 60, \ \ldots$

$T < 9 \Rightarrow \exists$ finite $\{T, V_{NA}, H\}$ [KMT]

general T: finite w/ some assumptions [Kim, Xu, Vafa '24]

3.7 7D SUGRA

7D SUGRA: het$/T^{43}$ $\to$ $\mathcal{M}_{19,3} = SO(19,3;\mathbb{Z})\backslash SO(19,3) / SO(19)\times SO(3)$

-or-

$M/K3$ geometric reduction

$\mathcal{M} = \mathcal{M}_{K3}$ characterized by $\Omega = x + iy$ cpx str.
$J = z$ Kähler form

$(x, y, z) \in H_2(3, 19)$, $x^2, y^2, z^2 > 0$.

$\Rightarrow \mathcal{M}$: choice of 3-plane in $\mathbb{R}^{19,3}$ $\Rightarrow \mathcal{M}_{19,3}$!

Witten '95: conjectured theories =, much evidence

3.8 elliptic K3 & F-theory.

$M_7 \to F_8$: take K3 elliptic, section $\mathbb{P}^{2,3,1}$ bdle over $\mathbb{P}^1$
(W. model)

$T^2 \to 0$

$R_1 \to 0 \Rightarrow M \to IIA$
$R_2 \to 0 \Rightarrow IIA \to IIB$ (T-duality)

W-model $f \in \mathcal{O}(-4K)$, $g \in \mathcal{O}(-6K)$ $\Rightarrow$ $\Delta \in \mathcal{O}(-12K)$
(Kodaira condition)
$\Rightarrow$ CY total space

fiber T^2 picks out $(1,1)$ directions in $\Gamma^{3,19}$

$\Rightarrow \mathcal{M}_{\text{ell } K3} = SO(2,18;\mathbb{Z})\backslash SO(2,18) / SO(2)\times SO(18)$

matching duality.

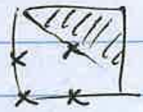
Complementary approaches to F-theory: IIB vs M-thy limit

3.6 K3

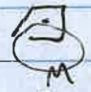
Only CY2 (cpx dim 2, $SU(2)$ holonomy; admits $R_{\mu\nu}=0$ metric, SUSY)
Kähler mfd. (comp. cpx str & metric)

$\pi_1(K3)=0$ (all loops top trivial)

$H_2(K3) = \Gamma^{3,19} = U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8)$
2d cpx $S \subset K3$, ($\partial S = 0$, $S \neq \partial V$) $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

ex. orbifold $T^4/\mathbb{Z}_2$  $(z_1, z_2) \to (-z_1, -z_2)$

ex. quartic in $\mathbb{P}^3$

[some alg. geometry: holomorphic line bundle  copy of $\mathbb{C}$ @ each $x \in M$
sections $\sim \psi(x)$ charged part. wf, possibly globally nontrivial, e.g. mag monopole in S^2

line bundle $\leftrightarrow$ divisor (codim 1 vanishing locus of $\mathcal{L}$) $\bigcirc \int F = 2\pi$
[alg. subvariety dim $d-1$]

$\Lambda^n T^*_x \Rightarrow$ line bundle, section $f\, dz^1 \wedge \ldots \wedge dz^n$
(f) divisor

Canonical class $K_X = [(f)]$

Ex. $K_{\mathbb{P}^n} = -(n+1)H$ ← hyperplane $z_{n+1}=0$

$K_{CY} = 0$ defining characteristic of CY.

Adjunction: X smooth variety, $S \subset X$ a smooth divisor

$$K_S = (K_X + S)\big|_S$$

so for $K_S = 0$, want $S = -K_X$!]

so if $S = 4H$ in $\mathbb{P}^3$, $X = CY2 = K3$.

I_n : n D7-branes $\Rightarrow$ SU(N) gauge gp.

F-theory hypothesis [Morrison/Vafa, Vafa] : each Kodaira type $\Rightarrow$ correspondg $\mathfrak{g}$!

Can "tune" various gauge groups $G \sim \Lambda_G \subset \Gamma^{18,2}$

e.g. $E_8 \times E_8$ $(u=0)$ $(u=1)$: $f = A u^4 (u-1)^4$ (deg 8)

$g = u^5 (u-1)^5 (B + Cu + Du^2)$ (deg 12)

4-param model

$E_6 \times E_6 \times E_6$: $g = u^4 (u-1)^4 (u-2)^4$

More subtle: tuning SU(N), SO(8+2n)

I_n: $[\Delta] = n$

Can solve order by order [MT] $f = -3a + (\,)u + \ldots$

$g = 2a + (\,)u + \ldots$

Or Tate form [Bershadsky et al]

$$y^2 + a_1 yx + a_3 y + x^3 + a_2 x^2 + a_4 x + a_6 \qquad \Delta$$

	a_1	a_3	a_2	a_4	a_6	Δ
deg	2	6	4	8	12	24
I_n	0	$\lfloor n/2 \rfloor$	1	$\lfloor \frac{n+1}{2} \rfloor$	n	n

3.5 geometric compactifications.

Assume $M_{10} = \mathbb{R}^{1,D-k-1} \times X_k$, D = 11 or 10

only metric g_{ij} $(+\phi)$ nontrivial, SUSY preserved

$\Rightarrow$ $\exists$ SUSY xform w/ $\delta\psi_\mu = D_\mu \eta = 0$ covariantly constant spinor

$\Rightarrow$ holonomy special, leaves cpt unchanged

(real) dim	mfd	holonomy	SUSY	
k	T^k	{1}	1	focus on these
4	K3 (CY2)	SU(2)	1/2	
6	CY3	SU(3)	1/4	
7	G_2	G_2	1/8	
8	Sp(2)	hyperKähler	3/16	
	CY4	SU(4)	1/8	
	Spin(7)	Spin(7)	1/16	

3.4 Nonabelian Gs in F-theory

Beautiful work of Kodaira: classify codim 1 singularities of ell fibrations. $\leftrightarrow$ ADE Dynkin diagrams (affine)

Singularity resolution:

blowing up a point $p \in \mathbb{C}^n$: replace w/ $\mathbb{CP}^{n-1}$

$p \longrightarrow \{\text{lines through } p\}$

lim $\to \in \mathbb{CP}^{n-1}$

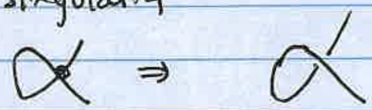
Ex $p \in \mathbb{C}^2$

$$\begin{array}{ll} B & \\ \downarrow \pi & \pi^{-1}(p \neq 0) = p \\ \mathbb{C}^2 = \{x,y\} & \pi^{-1}(p=0) \cong \mathbb{CP}^1 \end{array}$$

can describe in patches etc.

b-up smooths singularity

e.g.



Kodaira classification [e.g. Barth Hulek Peters, Van de Ven]

if $\begin{array}{c} X \\ \downarrow \\ D = \text{disc in } \mathbb{C} \end{array}$ elliptic fibration, $\pi^{-1}(u)$ smooth except @ $u = 0$

Classification of singular fibers $\longrightarrow$ Affine Dynkin diagrams

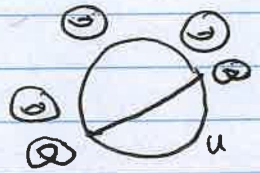
ord f	ord g	ord Δ	type	ADE Q.	Figure
0	0	0	nonsing	—	—
0	0	1	I_1	total space smooth	∞
0	0	n	I_n	$\hat{A}_{n-1}$	n ($\mathbb{P}^1$'s.)
1	1	2	II	total space smooth	
1	2	3	III	$\hat{A}_1$	
2	2	4	IV	$\hat{A}_2$	
2	3	6	I_0^*	$\hat{D}_4$	
2	3	6+n	I_n^*	$\hat{D}_{4+n}$	
3	4	8	IV^*	$\hat{E}_6$	
3	5	9	III^*	$\hat{E}_7$	
4	5	10	II^*	$\hat{E}_8$	
4	6	12	non resolvable	—	—

Standard Weierstrass form in $\mathbb{P}^{2,3,1}$ $\{y^2 = x^3 + f x(z^2) + g(z^4)\} \Rightarrow C$

[can write addition law in C] $\quad f, g \to j(\tau) = 1728 \cdot \frac{4f^3}{\Delta}$ $\to \tau$

Now fiber over $\mathbb{P}^1$

$$y^2 = x^3 + f(u) x + g(u)$$



Discriminant: $\Delta = 4f^3 + 27g^2$ vanishes @ singular points.
$[C = \partial_x C = \partial_y C = 0]$

$$C = -y^2 + x^3 + fx + g$$
$$\partial_y C = -2y$$
$$\partial_x C = 3x^2 + f$$

$$y = 0 \Rightarrow x^3 + fx + g = 3x^2 + f = 0$$
$$\hookrightarrow x^3 + fx/3 = 0$$
$$\Rightarrow \tfrac{2}{3} f x + g = 0 \Rightarrow x = -3g/2f \Rightarrow -\frac{27g^3}{8f^3} - \frac{g}{2} = 0$$
$$\Rightarrow \Delta = 0!$$

<u>8D F-theory</u>

Singular pts ~ 7-branes (carry $\pi/6$ defect)

$\Rightarrow f_8(u),\ g_{12}(u) \Rightarrow$ cpt S^2.
(elliptic)

$\pi: K3 \to B = 1$

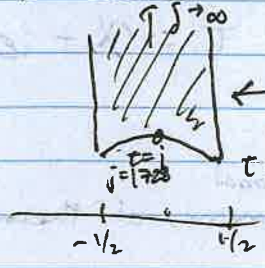
Total space: <u>K3 surface</u> (2D Calabi-Yau) $\pi^{-1}(x) \cong E$
[more later]

Generically, 24 separate singular points.

$$G = U(1)^{20}$$ (subtle: U(1) on branes gauged away, 20 U(1) in $B_{\mu\nu}$ [DPS])

What happens when singularities come together?

* $j(\tau)$ defined on $\mathbb{H}$ (upper half-plane), $SL(2,\mathbb{Z})$ invariant



← fund. domain

$\tau = e^{\pi i/3}$, $j=0$, $f=0$

can invert : $j(\tau) = \dfrac{1728}{4\alpha(1-\alpha)}$

$$\Rightarrow \tau = i \, \frac{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1 ; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1 ; \alpha\right)} \qquad (\tau, -1/\tau)$$

3.3 F-theory in 8D.

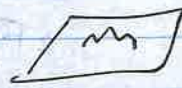
Consider IIB. Axiodilaton $\tau = \chi + i e^{-\phi}$

xforms under $SL(2,\mathbb{Z})$

$$\left.\begin{array}{l} T : \tau \to \tau + 1 \\ S : \tau \to -1/\tau \end{array}\right\} \text{ global symmetries of theory.}$$

10D IIB : 7-branes : charged objects w/ $SL(2,\mathbb{Z})$ monodromy of τ.

D7-brane : $\emptyset_{\tau \to \tau+1}$

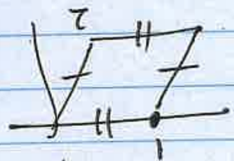
↳ dynamical quantum object  open strings ending on D7
→ dynamics (SYM on w-volume
N D7's ⇒ $U(N)$ gauge theory

As SUGRA solution, carries deficit angle $\frac{\pi}{6}$

Compactify IIB on S^2 ($\mathbb{CP}^1$).

 24 7-branes ⇒ total deficit 4π → compact

SUSY solution ?

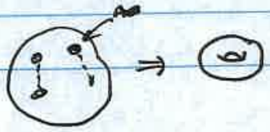
complex torus 

w/ marked point (add. ϕ) : elliptic curve

Alg geometric description (as variety = alg. hypersurf.) in $\mathbb{P}^2$

$$y^2 Z = X^3 + f X Z^2 + g Z^3$$

[Exercise : $y^2 z + a_1 xyz + a_3 y z^2 = x^3 + a_2 x^2 z + a_4 x z^2 + a_6 z^3$]
complete square, cube

idea: double cover of $\mathbb{P}^1$ 

3. N=1 8D SUGRA (16 supercharges)

3.1 8D ~~[illegible]~~ SUGRA

[Lee, Mikami '21]

gravity multiplet $\left(\underset{(20)}{g_{\mu\nu}}, \underset{(1)}{X}, \underset{(15)}{B_{\mu\nu}}, \underset{(12)}{A_\mu^{i=1,2}} \right)$

gauge multiplet $\left(\underset{(6)}{A_\mu}, \underset{(2)\ (\text{cpx})}{\phi} \right)$

signature $(2, \ell)$: ℓ gauge multiplets

Global anomalies $\Rightarrow \ell = 2, 10, 18$ (?) [Montero / Vafa '21]

[1710.04218 García-Etxebarria et al]

3.2 Heterotic cpt. (no orbifold) : $\ell = 18$

$\Gamma^{18,2} \subset \mathbb{R}^{18,2}$

(Narain) Moduli space: $\mathcal{M}_{18,2} = SO(18,2;\mathbb{Z}) \backslash SO(18,2;\mathbb{R}) / SO(18)\times SO(2)$

idea: move Γ w/ $SO(18,2)$; equivalences under $SO(18)\times SO(2)$, lattice automorphism...

G: $\Lambda_G \subset \Gamma^{18,2}$ Not just $\subset E_8 \times E_8$, ...

e.g. $A_{18} \subset \Gamma^{18,2} \Rightarrow SU(19)$ gr.

$D_{18} \longrightarrow SO(36)$

$E_6 \oplus E_6 \oplus E_6$

} From Nikulin thms.

Generically, $G = U(1)^{20}$ (2 from SUGRA multiplet; (18,2) sig)

Cont. Moduli: $\frac{20\cdot 19}{2} - \frac{18\cdot 17}{2} - 1 = 36 = 18$ cpx

$190 - 153 - 1$

$\square$ shape τ; 1 size g_{12} ; B_{12} } 2 cpx

$\hookrightarrow$ wilson lines $A_i \in \mathfrak{g}$ $\Rightarrow$ 16 cpx

Γ even $\leftrightarrow$ $v \cdot v \in 2\mathbb{Z}$ $\forall$ $v \in \Gamma$

$\underset{\text{(dual lattice)}}{\Gamma^*} = \{ \omega : \omega \cdot v \in \mathbb{Z} \ \forall v \in \Gamma \}$

Γ self-dual / unimodular $\Leftrightarrow$ $\Gamma = \Gamma^*$ $\Longleftrightarrow$ $\det \Gamma_{ij} = \pm 1$

Thm (Milnor) $\Gamma^{p,q}$ even unimodular $\Rightarrow$ $p \equiv q \pmod 8$

Ex's: Root lattices of all ADE algebras all even sig $(p,0)$
eg. $A_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ even, not unimodular (det = 3)

E_8: unique $(8,0)$ unimodular lattice

Back to heterotic string: L: 26-dim bosonic string
R: 10-d superstring

10D: cpt L on 16 dim even self-dual lattice
$\Rightarrow E_8 \times E_8$ or $\Lambda_{16} \approx \mathrm{Spin}(32)/\mathbb{Z}_2$!

9D: $\Gamma^{17,1}$

$(10-k)D$: $\Gamma^{16+k,k}$ $\quad$ l_L: $16+k$ dim real subspace of $\mathbb{R}^{16+k,k}$

$$V_L \qquad \qquad \cup \; \Gamma$$

$$G: \text{ root lattice } \Lambda_G = V_L \cap \Gamma^{16+k,k}$$

$\Rightarrow$ allowed gauge groups in dim $10-k$:

$\oplus$ ADE lattices $\subset \Gamma^{16+k,k}$
$\otimes$ $U(1)$'s, $rk = 16+2k$.

Also orbifolds: divide by discrete symmetry, can give $rk = 8+2k$, $2k$.

(ref. Nikulin: lattice embedding ~)

solution

$$X = x + w\sigma + p\tau + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{i}{n} \alpha_n e^{in(\tau+\sigma)} + \frac{i}{n} \tilde{\alpha}_n e^{in(\tau-\sigma)} \right)$$

$[\alpha_n, \alpha_m^+] = in\,\delta_{nm}$, $\alpha_m^+ = \alpha_{-m}$

$$= x + \frac{1}{\sqrt{2}} l_L(\tau+\sigma) + \frac{1}{\sqrt{2}} l_R(\tau-\sigma) + \ldots \qquad ({}^L/_R \text{ movers})$$

$$\left(l_L, l_R = \tfrac{1}{\sqrt{2}}(p \pm w)\right)$$

On S^1 : $w = mR$, $p = {}^n/_R$

Define $l \cdot l = l_L^2 - l_R^2 = 2nm \in 2\mathbb{Z}$.

$l = (l_L, l_R)$

$\Rightarrow l \in$ even lattice Γ (1D cpt, Γ'' has $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ i.p.)

in any D, consistency (modular invariance) $\Rightarrow l \in \Gamma$ even, self-dual

Quantum states of string $\prod \alpha_{-n} \tilde{\alpha}_{-m} | l_L, l_R \rangle$ $(\alpha, \tilde{\alpha} | l_L, l_R \rangle = 0)$

$\rightarrow$ particle in Poincaré rep, $M^2 = \frac{2}{\alpha'}\left[l_L^2 + 2(N-1)\right] = \frac{2}{\alpha'}\left[l_R^2 + 2(N-1)\right]$

massless fields: $(\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu \pm \alpha_{-1}^\nu \alpha_{-1}^\mu) | l_L = l_R = 0 \rangle$

$\Rightarrow g^{\mu\nu}, B^{\mu\nu}, \phi$

$\tilde{\alpha}_{-1}^\mu | l_R = 0, l_L^2 = 2 \rangle$

$\Rightarrow$ extra A_μ massless!

(e.g. $R = 1/R$, $m = n = \pm 1$, $l_L = \sqrt{2}$ $\rightarrow SU(2)$)

Lattices

(ref: Conway + Sloan)

$\Gamma \subset \mathbb{R}^{k,m}$ a lattice:

- $\Gamma = \{ \sum_{i=1}^{k} n_i e_i ,\ n_i \in \mathbb{Z} \}$ (e_i lin indep)
- $v \cdot w \in \mathbb{Z}\ \forall\, v, w \in \Gamma$ (integral, symmetric inner product)

Given basis, $\Gamma_{ij} = e_i \cdot e_j$ "Gram matrix" eg. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Eucl. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\Gamma^{1,1}$

2.2 Heterotic compactification on tori (16 Q's)

G: $E_8 \times E_8$, $Spin(32)/\mathbb{Z}_2$

G 10D g_{AB} B_{AB} A_A

ϕ, A'_μ, $g_{\mu\nu}$, A''_μ $B_{\mu\nu}$ A_μ $\underline{\Phi}$

e.g. $\underset{(10D)}{E_8 \times E_8} \longrightarrow E_8 \times E_8 \times U(1) \times U(1)$ (9D)

$E_8 \times E_8 \times U(1)^k \times U(1)^k$ (10-k D)

$rk\, G = 16 + 2k = 36 - 2D$

$\underline{\Phi}$: matter in adjoint rep.

But : • G can be smaller (rk fixed)

• G can be bigger

Flat connections $F_{\mu\nu} = 0 \Rightarrow$ locally $A_\mu \sim \emptyset$ by gauge xform

(break G)

$\pi_1(T^d)$ nontrivial $\Rightarrow$ flat cnxns $\left.\begin{array}{l} A_i = \text{const} \\ [A_i, A_j] = 0 \end{array}\right\} \Rightarrow F = \partial_\mu A_\nu - \partial_\nu A_\mu + i e[$

$U = e^{iA_i 2\pi R_i} = \oint e^{i\int dx^i A_i}$

breaks part of G. $g^{-1} U g \neq U$, g broken. $(m^2 A_\mu A^\mu$ - $i f(A_\mu, A_i$

A_i in Cartan $\Rightarrow$ maintains rank.

(ref: Polchinski)

String states: enhance G.

Simple bosonic string theory: $S \sim \frac{T}{2}\int d\sigma d\tau \left(\partial_\tau X \partial_\tau X - \partial_\sigma X \partial_\sigma X\right)$

EOM $(\partial_\tau^2 - \partial_\sigma^2) X = 0$

2. Compactifications: lattices, tori, ~~ellipsoids~~

2.1 Circle (S^1) compactification

Kaluza–Klein: 5D g_{AB} $A,B = 0,\dots,4$ $\downarrow$ 4D $A_\mu(g_{\mu 4})$, $\phi(g_{44})$, $g_{\mu\nu}$

EH action $\downarrow$ EM + U(1) YM

[Einstein: unify GR + EM]

11D SUGRA $\xrightarrow{S^1}$ 10D IIA

$g_{IJ} \to A_\mu, \phi, g_{\mu\nu}$

$C_{IJK} \to B_{\mu\nu}$, $C_{\mu\nu\lambda}$

membrane: wrapped $\to$ string; unwrapped $\to$ D2-brane.

[in more detail, 11D M-theory: strong coupling limit of IIA]
(decompact—) lots recent work.
($\phi = g_{nn}$, $g_s \sim e^{\phi}$)

* 10D IIA $\xrightarrow{S^1 \text{ rad } R}$ 9D (A_μ, $C_{\mu\dots}$)

10D IIB $\chi, B_{\mu\nu}, D^+_{\mu\dots}$ $\xrightarrow{S^1 \text{ rad } 1/R}$ 9D (really α'/R, $\alpha' = l_s^2$)

9D $\chi, A, B, C \dots$

$\Rightarrow$ Same theory! ("T-duality")

m/R momentum $\longleftrightarrow$ mR string winding

($\partial e^{imx/2\pi R} \sim \frac{m}{R}$)

Big picture:

11D: M $\dashrightarrow$ $S^1/\mathbb{Z}_2$ $\dashrightarrow$ $E_8 \times E_8$

10D: IIA, IIB, $E_8 \times E_8$, SO(32)

9D: II9 (IIA $\searrow$, IIB $\swarrow$); $E_8\times E_8$, SO(32) $\to$ same also…

32 Q's, unique in each D

4D: $\mathcal{N}=8$ SUGRA

(SM talk)

next: Lattices.

• 10D $\mathcal{N}=1$ SUGRA (type I SUGRA)

gravity multiplet
$8_v \otimes (8_v \oplus 8')$
of $SO(8)$

$$\underbrace{\underset{1}{\phi},\ \underset{\frac{8\cdot 9}{2}-1\ (35)}{g_{\mu\nu}},\ \underset{28}{B_{\mu\nu}}}_{64} \quad + \quad \underbrace{\underset{7\cdot 8}{\psi^{M}_{\alpha}},\ \underset{8}{\psi_{\alpha}}}_{64}$$

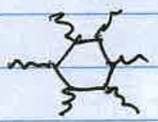
gauge multiplet
$(8_v \oplus 8')$

$$\underset{(8)}{A_\mu} \quad + \quad \underset{(8')}{\chi_\alpha}$$

gauge field
(can be NA)

- gravity ($g_{\mu\nu}$)
- $B_{\mu\nu} \to$ string theory
- $A_\mu \to$ gauge symmetry G.

Witten: Anomalies ($g_{\mu\nu}$, gauge, mixed) $\Rightarrow$ G broken @ quantum level
(dim $4k+2$)



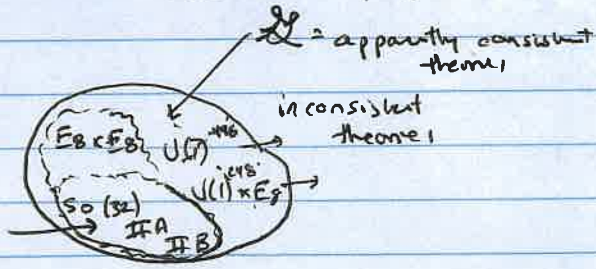
$\sim \partial_\mu j^\mu_5 \propto F\wedge F$ in 4D

Green & Schwarz: canceled$_{(\text{by } B)}$ for

$$G = \boxed{Spin(32)/\mathbb{Z}_2\ (\text{"}SO(32)\text{"}),\quad E_8\times E_8},\ \underbrace{\overset{248}{E_8\times U(1)},\ \overset{496}{U(1)}}_{\text{ADT! } \times \text{ inconsistent}}$$

heterotic string thy

Upshot:



~ 1984

now: in 10D $\mathcal{S} = \mathcal{Q}$: "string universality" (@ level of mass spectr
→ all consistent massless 10D SUGRA spectra from string thy.

Goal: Understand theories in lower D
• Consistency from susy, anomalies, etc.
$\boxed{\text{• SUGRA / string compactifications}}$ ← focus of remaining lectures.

[Gn ref:
WT TASI
notes]

So max dim for SUGRA is $D=11$, w/out spin >2 issues.

- unique ~~theory~~, massless spectrum: $g_{\mu\nu} + C_{\mu\nu\lambda} +$ ferm.
- 32 supercharges
- not a string theory (part. q couples to $\int_{w\text{-line}} A_\mu$, string to $\int_{w\text{-sheet}} B_{\mu\nu}$)
- "M-theory" ~ membrane thy $\left(\int_{w\text{-volume}} C_{\mu\nu\lambda}\right)$ [Q version: dHM, BFSS) [Victor's lectures]
- discovered 1978, unified w/ string thy 1995 [Witten]
- low-E action: $\frac{1}{2\kappa_{11}^2}\int\sqrt{-g}\left(R - \frac{1}{2}|G|^2 \mp \frac{1}{6} C\wedge G\wedge G\right)$, $G = dC$
  $[G_{\mu\nu\lambda\sigma} = \partial_{[\mu} C_{\nu\lambda\sigma]}]$

~~Goal: classify SUGRA thys in D≤11, connect w/ QFT/string physics~~

1.2.2 10D SUGRA

Using similar techniques:

- 10D $\mathcal{N}=2$ SUGRA ($32 = 2\times 16$ supercharges)

opp chirality: $\underline{16}, \underline{16}'$ $\Rightarrow$ IIA (parity invariant)

$$\underbrace{\phi,\ g_{\mu\nu},\ B_{\mu\nu}}_{64},\ \underset{\substack{(8)\\ [D\emptyset,}}{A_\mu},\ \underset{\substack{8\cdot 7\cdot 6/6\\(56)\\ D2]}}{C_{\mu\nu\lambda}} \Rightarrow 128 \text{ bosonic fields } (+128 \text{ fermions})$$

same chirality $\Rightarrow$ IIB

$$\phi,\ g_{\mu\nu},\ B_{\mu\nu},\ \underset{\substack{(1)\\ [D1],}}{\chi},\ \underset{(28)}{\tilde{B}_{\mu\nu}},\ \underset{\substack{(35)\\ D3]}}{D^{+}_{\mu\nu\lambda\sigma}} \Rightarrow 128+128 \text{ b+f DoF}$$

(self-dual)

IIA, IIB unique as massless spectra; both realized through quantum string theories $\int_\Sigma B_{\mu\nu}$ [Dp-branes: couple to p+1 fo

~~1.2 Supergravity~~

Enhanced SUSY in 4D: [SM lecture]

$$\{Q^A_\alpha, \bar{Q}^B_\beta\} = 2P_\mu \Gamma^\mu_{\alpha\beta}\delta^{AB} + Z^{AB}\delta_{\alpha\beta} \quad A \in 1, \ldots, N$$

each $\{b_A, b_A^\dagger\} = 1 \Rightarrow b^\dagger : \lambda \to \lambda + 1/2$

relates multiple $N=1$ multiplets $\Rightarrow$ 1 multiplet.

max helicity 2 $\Rightarrow$ max $N = 8$ $\quad |-2\rangle \cdots \quad |2\rangle$

Q's = "# of supercharges" $\Rightarrow N \times 4$ (32 max)

1.2 Supergravity

Look @ SUSY reps in higher dimensions.
problems when $\sim \lambda > 2$ in 4D, $\sim$ spin > 2 particles.

1.2.1 11D SUGRA

~~Effects~~ massless spin 2 $\to$ all spins.
[no known interacting massless higher spin theory]

Max D: $D = 11$

$$\{Q_\alpha, \bar{Q}_\beta\} = 2P_I \Gamma^I_{\alpha\beta} \quad I \in \{0, \ldots, 10\}$$

massless states xform under $SO(9)$ little group.

Exercise: 256 states in massless multiplet

44: $g_{\mu\nu}$ ($\square\square$ diag of $SO(9)$)

84: $C_{\mu\nu\lambda}$ ($\boxminus$, antisym 3-index tensor)

128: $\psi^M{}_\alpha$ (gravitino)

~~[illegible]~~ Puttin it all together

- construct spinor reps w/ creation & annihilator ops

$SO(N) \quad \{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$

$$\beta_a^+ \equiv \frac{1}{2}(\Gamma_{2a-1} + i\Gamma_{2a}) \qquad a = 1, \dots, n = \lfloor N/2 \rfloor$$
$$\beta_a = \frac{1}{2}(\Gamma_{2a-1} - i\Gamma_{2a})$$

$$\{\beta_a, \beta_b^+\} = \delta_{ab} \qquad \beta_a^2 = (\beta_a^+)^2 = 0 \qquad \text{(fermion c/a ops)}$$

$$\forall a \quad \beta|-\rangle = 0$$
$$\beta\beta^+|-\rangle = |-\rangle \Rightarrow \beta^+|-\rangle \equiv |+\rangle$$

State space $\quad V = (V_2)^{\lfloor N/2 \rfloor}$

(eg recall $SO(4) = SU(2) \otimes SU(2)$)

$$[M_{2a-1,2a}, \beta_a^+] = \beta_a^+ \qquad \Rightarrow \text{spin states } S_1, \dots, S_n \quad S_i \in \{\pm 1/2\}$$

$\uparrow$ S_a : cartan subalg.

- Back to 4D Mink. SUSY $\quad \Gamma^0 \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Gamma^1 \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on S_1

$$\beta_1 = \Gamma_0 + \Gamma_1 \qquad \beta_1^+ = \Gamma_0 - \Gamma_1 \qquad \left(\begin{matrix}\text{Mink} \\ \to \text{no } i!\end{matrix}\right)$$

$$\Gamma^1\Gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \mathbb{1} + \Gamma^1\Gamma^0 \quad \text{projects } S_1 = +1/2$$

$$\Rightarrow \{Q_{s_1,s_2}, Q^+_{s_1's_2'}\} = 2k\,\delta_{s_1,1/2}\,\delta_{s_2 s_2'} \quad \text{on massless } (k,k,0,0) \text{ states}$$

$\left[b = \frac{\sqrt{2}}{\sqrt{k}} Q\right. \qquad b, b^+ \propto Q, Q^+ ~~[illegible]~~ \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ raise & lower S_1.

$\Rightarrow$ SUSY relates helicity states $|\lambda\rangle, |\lambda + 1/2\rangle$

e.g. $|0\rangle, |1/2\rangle$ "sfermion", fermion
$|1/2\rangle, |1\rangle$ "gaugino", gauge boson
$|3/2\rangle, |2\rangle$ "gravitino", graviton

[note: in little gp picture all $\to$ 0,1 indices, no other $\pm$

1.1.2 Poincaré gp

SO(N): in natural / fund rep, M preserves δ_{ij} (Eucl. inner product)

$$M \delta M^T = \delta \qquad \delta = \begin{pmatrix} 1 & & \\ & 1_{ij} & \\ & & 1 \end{pmatrix}$$

In special relativity, $\eta_{\mu\nu} = \begin{pmatrix} +1 & & \\ & -1 & \\ & & \ddots \\ & & & -1 \end{pmatrix}$ (sig (1, d-1))

Lorentz xforms $\Lambda \eta \Lambda^T = \eta$ $\qquad \Lambda \in SO(1, d-1)$

<u>non-cpt</u> Lie gp, includes <u>boosts</u>, unitary reps ∞-dim.

in QFT $\Rightarrow$ Poincaré gp, include xlations

$$\left.\begin{aligned} J^{\mu\nu} &= i(x^\mu \partial^\nu - x^\nu \partial^\mu) \\ P^\mu &= i \partial^\mu \end{aligned}\right\} \text{Poincaré gens.}$$

$P^2 = P_\mu P^\mu$ commutes w/ all: diagonalize

- reps of Poincaré gp: classify particle states & fields (e.g. Weinberg
- uses "little group" method: fix p^μ. look @ rep. e.g. massive $p = (p_0, \ldots) \to$ spin $SO(d-1)$; massless $(p, p, 0, 0) \to$ helicity $SO(d-2)$ ← $E(d-2)$ → spin

1.1.3 SUSY

extend Poincaré $\Rightarrow$ <u>super</u> Poincaré: introduce Q_α (spinor index)

Q_α: boson $\longleftrightarrow$ fermion

$$\{ Q_\alpha, \bar{Q}_\beta \} = 2 P_\mu \Gamma^\mu_{\alpha\beta}$$

Ex: 4D N=1 SUSY

$$\{ \Gamma^\mu, \Gamma^\nu \}_{\alpha\beta} = 2 \eta^{\mu\nu} \mathbb{1}_{\alpha\beta}$$

$$\bar{Q} \equiv Q^\dagger \Gamma_0$$

$\Rightarrow$ little gp massless $p^\mu = (k, k, 0, 0) \to \{ Q_\alpha, Q_\beta^\dagger \} = 2k (\mathbb{1} + \Gamma^1 \Gamma^0)_{\alpha\beta}$

Thm (Dynkin) • highest wt labeled by $\mu_a \to (n_1, \dots n_r)$

• state w/ $n_a > 0$ hit n_a times by $e_{-\alpha_a}$ $\Rightarrow$ new state, $\to$ subract a^{th} row of C_{ab}

• $\prod_{k=1}^{\ell} e_{-\alpha_a} \mu$ @ level ℓ.

• reps "spindle shaped" $\langle\rangle$

e.g. $SU(3)$ $\quad C_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\mu = (2, 0)$

$\downarrow e_{-\alpha_1}$

$(0, 1)$

$e_{-\alpha_1} \swarrow \quad \searrow e_{-\alpha_2}$

$(-2, 2) \qquad (1, -1)$

$e_{-\alpha_1} \searrow \quad \swarrow e_{-\alpha_2}$

$(-1, 0)$

$\downarrow$

$(0, -2)$

$\square\!\square$ of $SU(3)$

$\leftrightarrow$ symm 2-index...

Exercise $SO(8)$ fund. rep

$1\,0\,0\,0$

$\Downarrow$

$-1\,1\,0\,0$

$\vdots$

Spinor reps of $SO(N)$ $\quad (N \geq 5)$

• important for physics + SUSY

• only reps not in $\Lambda \square$ for classical alg's.

$SO(2N)$ $\quad$ o—o—o$\cdots$ with fork to two nodes 2^{N-1}, 2^{N-1} $\qquad 0\,0\,0\,{}^{1}_{0}$

dims of $\underline{1}$ reps (fund) $\quad 2N \quad \binom{2N}{2} \dots$

$SO(2N+1)$ $\quad$ o—o—o$\cdots$o$\Leftarrow$o $\quad 2^N$

$2N+1 \quad \binom{2N+1}{2} \quad \dots$

realization: use Clifford algebra $\{\Gamma_i, \Gamma_j\} = \Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij} \mathbb{1} \quad i,j = 1,\dots,N$

$M = \frac{1}{4i} [\Gamma_i, \Gamma_j] \to SO(N)$ alg, realize spinor rep.

e.g. $N = 5$ Dirac matrices $\Gamma_1, \dots \Gamma_4$ 4×4, $\Gamma_5 \propto \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$

- ADD from affine (∞-dim) Lie algs.

- $M_{ADD} \not\subset M_{\text{Lie alg}}$ $\Rightarrow$ classificat (pf: <u>exercise</u>)

DD, ADD useful in understanding

- equivalences e.g. ○ ○ $SO(4) \sim SU(2) \times SU(2)$
  ○—○—○ $SO(6) \sim SU(4)$

- outer automorphisms of $\mathfrak{g}$ (from symmetries of DD)

- classify subgroups, reps.

<u>Representations</u> ~~Rees~~ rep $g \to \mathcal{D}(g) : \mathcal{H} \to \mathcal{H}$ assume unitary reps.
gp. hom

~~Classification:~~ alg: $h \to \mathcal{D}(h)$ $(g \sim 1 + i\varepsilon h)$

reducible: $\mathcal{D} \sim \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}$ $\forall g$; focus on irred. reps.

Classification

- diagonalize $\mathfrak{g}_0$: $h_i |\mu\rangle = \mu_i |\mu\rangle$

  μ_i : <u>weights</u> (α wts in adjoint rep)

  $h_i (e_\alpha |\mu\rangle) = (\mu_i + \alpha_i)(e_\alpha |\mu\rangle)$

  $\Rightarrow$ $e_{\pm\alpha}$ move in weight space.

  $2 \dfrac{\mu \cdot \alpha_i}{\alpha^2} \in \mathbb{Z}$ ($\pm\alpha \Rightarrow SU(2)$ subalg.)

- Finite dim reps R labeled by highest wt $|\mu_R\rangle$, $e_\alpha |\mu_R\rangle = 0$
  α simple

  labeled uniquely by $\Gamma_a = \dfrac{2 \mu \cdot \alpha_a}{\alpha_a^2} \geq 0$

- all states $\sim \prod e_{-\alpha} |\mu_R\rangle$; multiplicity a bit complicated.

~~Details Georgi~~

Cartan Matrix $C_{ab} = 2\frac{\alpha_a \cdot \alpha_b}{\alpha_a \cdot \alpha_a} \in \mathbb{Z}$ (pf, details Georgi)

e.g. $SU(3)$ $2\frac{\beta \cdot (-\delta)}{\beta^2} = -1$, $C_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

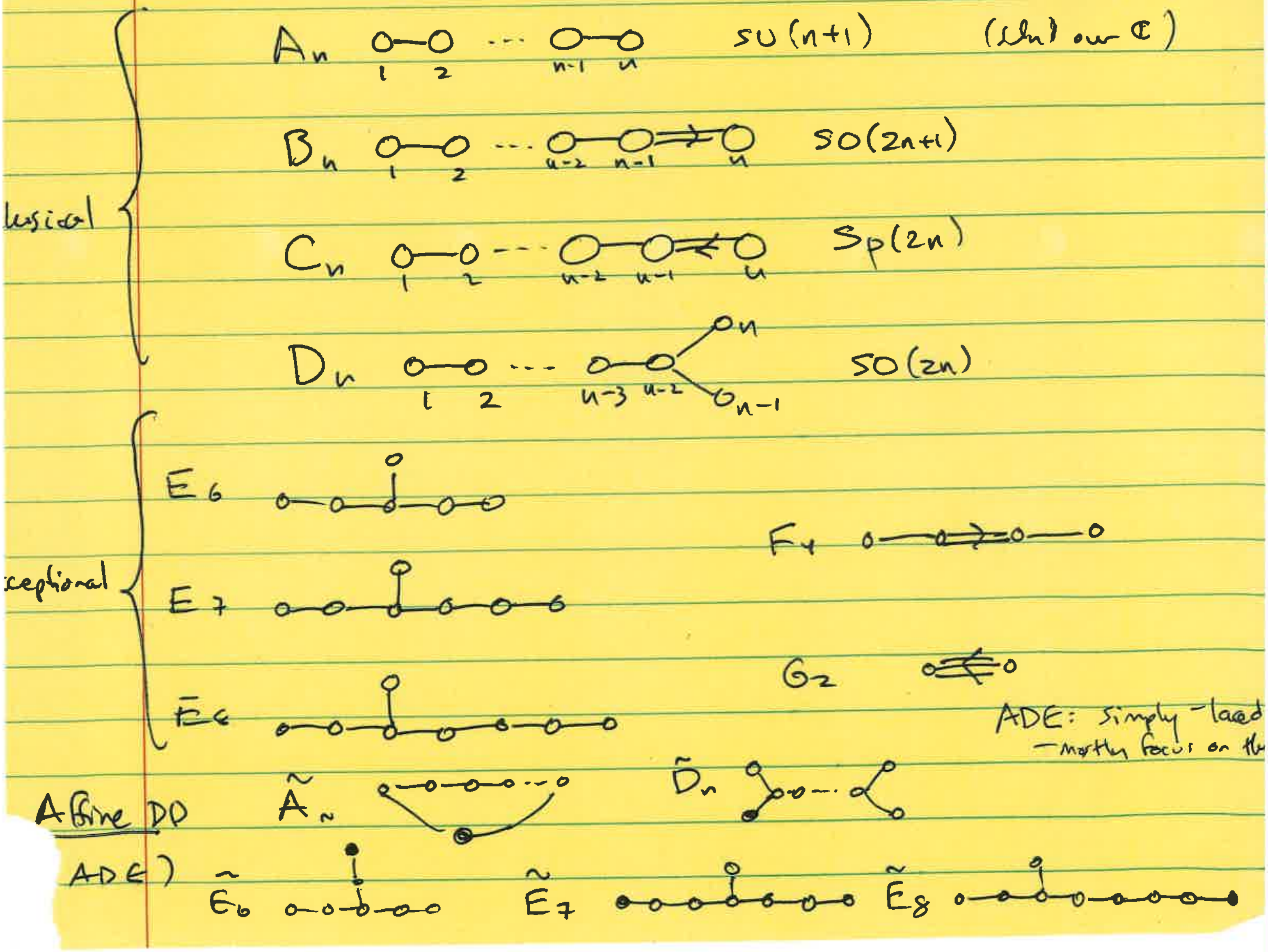
All (finite-dim) simple cpt Lie algebras:

sl_n, so_n, sp_{2n} (Classical)

E_6, E_7, E_8, F_4, G_2 (Exceptional algs)

Dynkin diagrams: $\frac{4(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \begin{cases} 0 & 90^\circ \\ 1 & 120^\circ \\ 2 & 135^\circ \\ 3 & 150^\circ \end{cases}$

short root



Cpt Lie groups, algebras & reps   ref. Georgi, Slansky

Group: $G = \{g\}$ $gh \in G$, $(gh)j = g(hj)$, $\exists 1 : 1 \cdot g = g \cdot 1$, $\forall g \exists g^{-1}$ $gg^{-1} = g^{-1}g$
[closed]   [assoc.]   (id)   (inverse)

Lie gp: G is a manifold (locally $\cong \mathbb{R}^n$)
ex. $SU(N)$ $UU^{+} = \mathbb{1}$, $\det U = 1$

Lie algebra: local structure of Lie gp
$G \to \mathfrak{g}$

$g \sim 1 + i\epsilon A$
$[A, B]$: closed, linear in A, B, antisymmetric $[A,B] = -[B,A]$
Jacobi $[[A,B],C]$ + cyclic $= 0$
ex. $\mathfrak{su}(2)$ $[J_a, J_b] = i\epsilon_{abc}J_c$ $\epsilon_{123} = 1$, antisymm.

Physics: Lie gps $\leftrightarrow$ gauge symm (+global) $(U(1), SU(2), SU(3))$.
Note: multiple $G \to$ same $\mathfrak{g}$ (e.g. $\mathfrak{su}(2)$ $\swarrow SU(2)$ $\nwarrow SO(3)$ $(\cong SU(2)/\mathbb{Z}_2)$)
matter: in representation of G, $\mathfrak{g}$
(rep $G \to$ rep $\mathfrak{g}$ $\Rightarrow$ suff to classify reps of $\mathfrak{g}$ + easier)

- Classification of cpt. simple Lie algebras (simple: nonabelian, no factors [no nonzero proper ideals])
(cpt $\Rightarrow$ all irreps finite dim)

i) Cartan subalgebra $[h_i, h_j] = 0$ (irreducible reps, no $\begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}$)
$\mathfrak{g}_0$ spanned by $h_1 \ldots h_m$ $i = 1, \ldots, m$ (m max) $\uparrow$ rank

ii) Diagonalize in adjoint rep. $[h_i, e_\alpha] = \alpha_i e_\alpha$ $\underline{\alpha}$: root $\in$
positive ~~simple~~ roots: $\alpha = (0, 0, \ldots, 0, \alpha_i > 0, \ldots)$
simple roots: $\alpha > 0$, $\alpha \neq \beta + \delta$ $\beta, \delta > 0$
ex. $SU(3)$  $\beta, -\delta$ simple roots.

1. Tools + 11D/10D SUGRA

1.1 Symmetries

QG is a hard problem. ⇒ Add symmetry!

4D Poincaré (rotations, boosts, xlations)

Basic idea: ⟶ 10D/11D Poincaré

⟶ 10D/11D Super Poincaré [will discuss why not $D \leq 11$]

Relevant symmetries:

- Gauge symmetries: cpt Lie groups (also relevant for F-theory geom
- Poincaré symmetry
- Supersymmetry

1.1.1 Compact

4th force: gravity – <u>not just a QFT</u>!

– GR (+SM) perturbatively non-renormalizable
– symmetry: diffeomorphism invariance
$\Rightarrow$ need to $\int$ over <u>all geometries, topologies</u>
! no clear mathematical framework in general

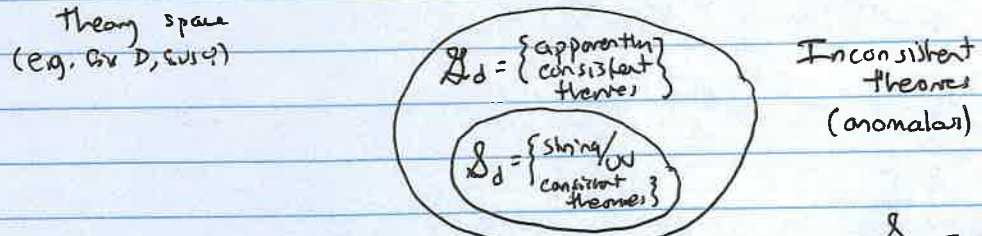
Only robust solution in higher dim: "<u>string theory</u>"

– Not just a theory of strings.
– 11D Sugra: membranes
– AdS/CFT: QG = CFT in specific asymptotic geometries

(Super) string theory: natural definition in 10D ($\to$ 11D)
$\to$ SUGRA is low-energy EFT.
$\Rightarrow$ theories in lower dim. by "compactification"

$$X_{10} = \underset{(\text{compact})}{M^{c}_{10-d}} \times \underset{(\text{spacetime: } \mathbb{R}^{1,d-1} \text{ or } AdS_d)}{M^{s}_{d}}$$

String theory / Quantum Gravity imposes constraints on Low E physic.
(cf. Montero lectures)



$\mathcal{S}_d = \mathcal{S}_d$?
<u>string universality</u>

These lectures: Explore $\mathcal{S}_d$ through geometric compactifications of SUGRA
– Focus on tools, principles. (little string theory)

Supergravity and string / F-theory compactifications in Various Dimensions

W. Taylor lectures @ TSIMF, January 2025
19th Asian ~~Pacific~~ Winter School on Strings, Particles, and Cosmology

0. Intro
1. Basic tools, SUGRA in 11D, 10D
2. Compactification
3. 8D & 7D SUGRA & compactification
4. 6D $N=1$ SUGRA & cpt.
5. 4D $N=1$ F-theory & outlook

0. Intro

QFT describes the observed Standard Model of particle physics

- Gauge fields A_μ, group $G = \underset{\uparrow \text{strong}}{SU(3)} \times \underbrace{SU(2)\times U(1)}_{\text{electroweak}} / \mathbb{Z}_6$
  $\Rightarrow \gamma, g, Z, W\pm$ bosons
- Matter: quarks $(u, d, \dots)$, leptons $(e^-, \nu_e, \dots)$ fermions, transform under G (represen
- Higgs field ϕ: scalar field, xforms under $SU(2)\times U(1)$
- $$\mathcal{L} \sim \int -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi + \bar{\psi}\phi\psi - V(\phi)$$
  [schematic] ($F_{\mu\nu} = \partial_\mu A_\nu + ie[A_\mu, A_\nu]$) (kinetic, chiral) (Yukawa)
- many details: chiral matter, SSB, etc..